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A PARAMETRIC CONVEX OPTIMAL CONTROL PROBLEM FOR A LINEAR SYSTEM†

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The dependence of the solutions of a terminal optimal control problem on a parameter in the initial state vector is investigated. Attention is devoted mainly to the behaviour of the solution in the neighbourhood of a non-regular point. On the basis of the results, a method is proposed for constructing solutions of the problem for all parameter values.

In practical work it is often important to know not only the solution of an optimal control problem for fixed parameter values, but also the dependence of the solution on the parameters, which enables one to estimate how the solution may vary when the parameters fluctuate. In addition, a knowledge of the dependence of the solutions of optimal control problems on the parameters provides the basis for methods of constructing feedback controls [1, 2], as well as stabilization and estimation methods based on the moving horizon strategy [3–5].

Numerical solutions of such problems are generally achieved by continuation of the solution with respect to a parameter [6–9]. The greatest difficulties in applying such methods arise in the case when the "actual" value of the parameter is a non-regular point. Therefore, in most publications devoted to sensitivity analysis and to investigating the parameter-dependence of the solutions, it is assumed that all parameter values are regular, or of degree of non-regularity one. In this paper no such assumptions are made.

1. FORMULATION OF THE PROBLEM

In the class of piecewise-continuous functions, we consider the family of terminal optimal control problems for a linear system whose initial states depend on a parameter $\tau \in [\tau_*, \tau^*]$

$$\mathrm{OC}(\tau): \begin{cases} f_0(x(t^*)) \to \min \\ \dot{x} = Ax + bu, x(0) = z(\tau) \\ f(x(t^*)) \leqslant 0, |u(t)| \leqslant 1, t \in T = [0, t^*] \end{cases} \tag{1.1}$$

$$A \in R^{n\times n},\ \ b \in R^n,\ \mathrm{rank}(b, Ab, \dots, A^{n-1}b) = n;\ \ f(x) = (f_i(x), i = 1, 2, \dots, m)$$

where $x = x(t) \in R^n$ is the state vector, $u = u(t) \in R$ is the control, $f_i(x)$ $(i = 0, 1, \dots, m)$ are continuously differentiable convex functions with bounded continuous derivatives $\partial^2 f_i(x)/\partial x_s \partial x_j$ $(s = 1, 2, \dots, n;$ $j = 1, 2, \dots, n; i = 0, 1, \dots, m)$ and $z(\tau)$, $\tau \in [\tau_*, \tau^*]$ is a given piecewise-smooth n-vector-valued function.

It is required to investigate how the value of the performance criterion and the solutions of problem OC(τ) depend on the parameter τ and to describe a rule for constructing solutions when the parameter is perturbed.

In what follows we shall assume that for any $\tau \in [\tau_*, \tau^*]$, problem OC($\tau$) satisfies Sleiter's condition: a control, $|\bar{u}_\tau(t)| \leqslant 1, t \in T$, exists such that on the corresponding trajectory $\bar{x}_\tau(t), t \in T$, the inequalities $f(\bar{x}_\tau(t^*)) < 0$ are satisfied.

Let $X(\tau)$ denote the set of attainability in problem OC(τ) and let X_0 be the set of solutions of the problem

$$f_0(x) \to \min,\ \ f(x) \leqslant 0,\ \ \|x\| \leqslant C_0$$

where $C_0 > 0$ is a fairly large number. We shall assume that for any $\tau \in [\tau_*, \tau^*]$ it is true that $X_0 \cap X(\tau) = \varnothing$.

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2. STRUCTURE OF THE SOLUTION. DEFINING ELEMENTS

Let us consider problem OC(τ) for fixed τ. Let $u_\tau(\cdot) = (u_\tau(t), t \in T), x_\tau(\cdot) = (x_\tau(t), t \in T)$ denote an optimal control and the corresponding trajectory, and let $\psi(t, y, x), t \in T, y \in R^m, x \in R^n$ be the solution of the adjoint system

$$\dot{\psi} = -A'\psi, \ \ \psi(t^*) = -\frac{\partial f_0(x)}{\partial x} - \frac{\partial f(x)}{\partial x} y \tag{2.1}$$

According to the maximum principle [10], a necessary and sufficient condition for a control $u_\tau(\cdot)$ in problem OC(τ) to be optimal is the existence of a vector $y \in R^m$ such that

$$y \geqslant 0, \ \ y'f(x_\tau(t^*)) = 0 \tag{2.2}$$

$$\psi'(t, y, x_\tau(t^*))bu_\tau(t) = \max_{|u| \leqslant 1} \psi'(t, y, x_\tau(t^*))bu, \ \ t \in T$$

Let $Y(\tau) \subset R^m$ denote the set of all vectors y possessing properties (2.1) and (2.2). Consider the point-set mapping

$$\tau \to Y(\tau), \tau \in [\tau_*, \tau^*] \tag{2.3}$$

Basing oneself on known results [11, 12], one can show that, under the above assumptions, the mapping (2.3) possesses the following properties:

1) for any $\tau \in [\tau_*, \tau^*]$, the set $Y(\tau)$ is non-empty and bounded;
2) the mapping (2.3) is upper semi-continuous;
3) for any convergent sequence

$$\{\tau_k\}_{k \to \infty}, \tau_k \in [\tau_*, \tau^*], \ \lim_{k \to \infty} \tau_k = \tau_0 + 0$$

the limit y^* of any convergent subsequence of the sequence $\{y_k\}_{k \to \infty}, y_k \in Y(\tau_k)$ is a solution of the problem

$$\psi'(0, y, x_{\tau_0}(t^*))\dot{z}(\tau_0 + 0) \to \min, \ \ y \in Y(\tau_0) \tag{2.4}$$

Let y be an arbitrary vector in $Y(\tau)$. Put

$$M = \{1, \ldots, m\}, \ \ M_a(\tau) = \{i \in M : f_i(x_\tau(t^*)) = 0\}$$

$$\{t_j(\tau), j = 1, 2, \ldots, p(\tau)\} = \{t \in T : \psi'(t, y, x_\tau(t^*))b = 0\} \tag{2.5}$$

$$t_j(\tau) < t_{j+1}(\tau), j = 1, 2, \ldots, p(\tau) - 1$$

Let us assume that $p(\tau) = 0$ for $\{t \in T\colon \psi'(t, y, x_\tau(t^*))b = 0\} = \varnothing$.

Let $q_j (j = 1, 2, \ldots, p(\tau))$ be numbers such that

$$W^{(q)}(t_j(\tau)) = 0, \ \ q = 0, 1, \ldots, q_j - 1; \ \ W^{(q_j)}(t_j(\tau)) \neq 0 \tag{2.6}$$

where $W^{(q)}(t) = \partial^q \psi'(t, y, x_\tau(t^*))b/\partial t^q (q = 1, 2, \ldots)$. Form the matrix

$$\Phi(\tau, y) = \left\| \begin{matrix} \dfrac{\partial f_i'(x_\tau(t^*))}{\partial x} a^{(q)}(t_j(\tau)), q = 0, 1, \ldots, q_j - 1; j = 1, 2, \ldots, p(\tau) \\ i \in M_a(\tau) \end{matrix} \right\|$$

$$(a(t) = F(t)b, a^{(q)}(t) = d^q a(t)/dt^q, \ \ \dot{F}(t) = -F(t)A, F(t^*) = E)$$

Both here and below it will be assumed that sets of subscripts written in the form $j = 1, 2, \ldots, p$ or $\{1, 2, \ldots, p\}$ are empty if $p = 0$, and that

$$\operatorname{rank} \left\| \begin{matrix} a_{ij}, j = 1, 2, \ldots, p \\ i \in P \end{matrix} \right\| = 0 \text{ if } p = 0 \text{ or } P = \varnothing.$$

Among the vectors $y \in Y(\tau)$ there is one such that

$$\operatorname{rank} \Phi(\tau, y) = |M_a(\tau)| \tag{2.7}$$

Indeed, let us suppose that for a given vector $y \in Y(\tau)$ we have rank $\Phi(\tau, y) = k < |M_a(\tau)|$. Then a vector $\Delta y \in R^m$, $\Delta y \neq 0$ exists such that

$$\left(\sum_{i \in M_a(\tau)} \frac{\partial f_i'(x_\tau(t^*))}{\partial x}\Delta y_i + \frac{\partial f_0'(x_\tau(t^*))}{\partial x}\right) a^{(q)}(t_j(\tau)) = 0$$
$$q = 0, 1, \ldots, q_j - 1; j = 1, 2, \ldots, p(\tau) \tag{2.8}$$
$$\Delta y_i = 0, \; i \in M \setminus M_a(\tau)$$

It follows from (2.8) that $y + \sigma \Delta y \in Y(\tau)$ for all sufficiently small $\sigma > 0$. Let σ_0 denote the largest $\sigma > 0$ such that $y + \sigma\Delta y \in Y(\tau)$. Since the set $Y(\tau)$ is bounded and closed, we have $\sigma_0 < \infty$. Put $\bar{y} = y + \sigma_0 \Delta y$. By construction, $\bar{y} \in Y(\tau)$. Let $\bar{t}_j(\tau), \bar{q}_j (j = 1, 2, \ldots, \bar{p}(\tau))$ denote the times (2.5) and numbers (2.6) corresponding to the vector $\bar{y}$. By construction, rank $\Phi(\tau, \bar{y}) >$ rank $\Phi(\tau, y)$. Consequently, in at most $|M_a(\tau)| - k$ steps one can find a vector $\bar{y} \in Y(\tau)$ for which (2.7) is true.

In what follows we let $y(\tau)$ denote vectors in the set $Y(\tau)$ that possess property (2.7). It was shown above that the set of such vectors is not empty.

Suppose $y(\tau) \in Y(\tau)$. Let $t_j(\tau), q_j(j = 1, 2, \ldots, p(\tau))$ denote the times (2.5) and numbers (2.6) corresponding to the vector $y(\tau)$. Put

$$M_0(\tau) = \{i \in M_a(\tau) : y_i(\tau) = 0\}$$

$$L(\tau) = \{j \in \{1, \ldots, p(\tau)\} : q_j > 1\}, \;\; k(\tau) = u_\tau(+0)$$

$$l_*(\tau) = 1 \text{ if } t_1(\tau) = 0, \; l_*(\tau) = 0 \text{ if } t_1(\tau) > 0 \text{ or } p(\tau) = 0$$

$$l^*(\tau) = 1 \text{ if } t_{p(\tau)}(\tau) = t^*, \; l^*(\tau) = 0 \text{ if } t_{p(\tau)}(\tau) < t^* \text{ or } p(\tau) = 0$$

We shall call the parameter sets

$$S(\tau) = \{p(\tau), k(\tau), M_a(\tau), l_*(\tau), l^*(\tau), M_0(\tau), L(\tau)\}$$

$$Q(\tau) = \{t_j(\tau), j = 1, 2, \ldots, p(\tau); y(\tau)\}$$

the structure and defining elements (corresponding to the vector $y(\tau)$) of the solution, respectively.

We shall say that a solution $u_\tau(\cdot)$ is non-degenerate (the value of the parameter τ is a regular point) if

$$\beta(\tau) = |M_0(\tau)| + l_*(\tau) + l^*(\tau) + |L(\tau)| = 0$$

It will be shown below that the concept of the non-degeneracy of solutions characterizes the stability of the structure of the solution to small variations of the parameter τ.

We shall prove a number of auxiliary propositions.

Lemma 1. Let $y(\tau) \in Y(\tau)$. If $\beta(\tau) = 0$, then $|Y(\tau)| = 1$.

Proof. Let $t_j(\tau), q_j(j = 1, 2, \ldots, p(\tau))$ denote the instants (2.5) and numbers (2.6) corresponding to the vector $y = y(\tau)$. Recall that, by agreement, we are assuming that $y(\tau)$ satisfies condition (2.7).

Let us suppose first that $p(\tau) = 0$. Then it follows from (2.7) that $|M_a(\tau)| = 0$. It follows from the second condition of the relations representing the maximum principle (2.2) that $Y(\tau) = y(\tau) = 0$. This proves the lemma for the case when $p(\tau) = 0$.

Now suppose $p(\tau) \geqslant 1$. It follows from the equality $\beta(\tau) = 0$ that

$$q_j = 1, j = 1, 2, \ldots, p(\tau), l_*(\tau) = l^*(\tau) = 0 \tag{2.9}$$

and condition (2.7) then becomes

$$\operatorname{rank}\left\| \begin{matrix} \frac{\partial f_i'(x_\tau(t^*))}{\partial x} a(t_j(\tau)), j=1,2,\dots,p(\tau) \\ i \in M_a(\tau) \end{matrix} \right\| = |M_a(\tau)| \tag{2.10}$$

It follows from relations (2.9) that all the times $t_j(\tau)$ ($j = 1, 2, \dots, p(\tau)$) are switching points of the control $u_\tau(\cdot)$. Taking this into account together with (2.2), we conclude that for any vector $y \in Y(\tau)$ necessarily

$$\left(\sum_{i \in M_a(\tau)} \frac{\partial f_i'(x_\tau(t^*))}{\partial x} y_i + \frac{\partial f_0'(x_\tau(t^*))}{\partial x} \right) a(t_j(\tau)) = 0, \quad j = 1, 2, \dots, p(\tau); \quad y_i = 0, \, i \in M \setminus M_a(\tau)$$

By condition (2.10), this system cannot have more than one solution, that is, $|Y(\tau)| \leqslant 1$. Since $Y(\tau) \neq \varnothing$, this means that $|Y(\tau)| = 1$. The lemma is proved.

Lemma 2. The property of non-degeneracy (degeneracy) of a control $u_\tau(\cdot)$ is independent of the choice of the vector $y(\tau) \in Y(\tau)$.

Proof. Suppose $y(\tau) \in Y(\tau)$ is such that $\beta(\tau) = 0$. Then, by definition, the control $u_\tau(\cdot)$ is non-degenerate, and, by Lemma 1, $y(\tau) \in Y(\tau)$. It is obvious that in that case there is no other vector $\bar{y}(\tau) \in Y(\tau)$ for which $\bar{\beta}(\tau) \neq 0$.

Now suppose $\beta(\tau) \neq 0$. By definition, the control $u_\tau(\cdot)$ is degenerate. Suppose a vector $\bar{y}(\tau) \in Y(\tau)$ satisfying condition (2.7) exists such that $\bar{\beta}(\tau) = 0$. But then, by Lemma 1, $|Y(\tau)| = 1$, which contradicts the assumption $y(\tau) \in Y(\tau), \bar{y}(\tau) \in Y(\tau), \bar{y}(\tau) \neq y(\tau)$. This contradiction completes the proof of Lemma 2.

Lemma 2 implies that the concepts of non-degenerate control and regular point are well defined.

3. PROPERTIES OF THE SOLUTIONS IN THE NEIGHBOURHOOD OF A REGULAR POINT

Suppose that for some parameter value $\tau_0 \in [\tau_*, \tau^*]$ we have a known optimal control $u_{\tau_0}(\cdot)$ of problem $OC(\tau_0)$ and a vector $y(\tau_0) \in Y(\tau_0)$. Let $S(\tau_0)$ and $Q(\tau_0)$ denote the structure and defining elements, respectively, corresponding to the vector $y(\tau)$, and $T^+(\tau_0)$ is a sufficiently small right neighbourhood of the point τ_0. We wish to investigate the properties of the solutions of problems $OC(\tau)$ for $\tau \in T^+(\tau_0)$.

Let us assume that τ_0 is a regular point. In that case, by Lemma 1, the set $Y(\tau_0)$ consists of a single element, and by property 2 of the mapping (2.3) we have $y(\tau_0 + 0) = y(\tau_0)$. Taking this into consideration, by analogy with the results of [11], one can show that for $\tau \in T^+(\tau_0)$ the parameters $S(\tau)$ and $Q(\tau)$ are uniquely defined by the relations

$$\begin{aligned} & S(\tau) = S, f_i(\mu(\tau, k, t_j(\tau), j = 1, 2, \dots, p)) = 0, i \in M_* \\ & y_i(\tau) = 0, i \in M \setminus M_* \\ & c'(\mu(\tau, k, t_j(\tau), j = 1, \dots, p), y(\tau))\, a(t_j(\tau)) = 0, j = 1, 2, \dots, p \end{aligned} \tag{3.1}$$

where

$$\mu(\tau, k, t_j, j = 1, 2, \dots, p) = F(0)z(\tau) + \sum_{j=0}^{p} k(-1)^j \int_{t_j}^{t_j+1} a(t)dt, \quad t_0 = 0, \quad t_{p+1} = t^*$$

$$c'(x, y) = \partial f_0'(x)/\partial x + y'\partial f'(x)/\partial x$$

$$p = p(\tau_0), \quad k = k(\tau_0), \quad S = S(\tau_0), \quad M_* = M_a(\tau_0)$$

and by the initial conditions

$$t_j(\tau_0 + 0) = t_j(\tau_0), \quad j = 1, 2, \dots, p; \quad y(\tau_0 + 0) = y(\tau_0) \tag{3.2}$$

The optimal control $u_\tau(\cdot)$ in problem $OC(\tau)$, where $\tau \in T^+(\tau_0)$, has the form

$$\begin{aligned} & u_\tau(t) = (-1)^j k, t \in [t_j(\tau), t_{j+1}(\tau)[, j = 0, 1, \dots, p \\ & t_0(\tau) \equiv 0, t_{p+1}(\tau) \equiv t^* \end{aligned} \tag{3.3}$$

Thus, if τ_0 is a regular point, the solutions of problems OC(τ) in its neighbourhood are uniquely defined by relations (3.1)–(3.3).

4. THE CONSTRUCTION OF SOLUTIONS IN THE NEIGHBOURHOOD OF A NON-REGULAR POINT

Now, assuming that τ_0 is not a regular point, let us investigate the properties of the solutions of problems OC(τ) for $\tau \in T^+(\tau_0)$. Note that now the set $Y(\tau_0)$ may consist of more than one vector, so that in the general case $y(\tau_0 + 0) \neq y(\tau_0)$. In addition, at $\tau = \tau_0$ the structure of the solution changes: $S(\tau_0 + 0) \neq S(\tau_0)$.

Thus, in order to construct solutions of problems OC(τ) for $\tau \in T^+(\tau_0)$, one has first to determine the vector $y(\tau_0 + 0)$ and parameters $S(\tau_0 + 0)$, $Q(\tau_0 + 0)$.

Construction of the vector $y(\tau_0 + 0)$. It follows from property 3 of mapping (2.3) that the vector $y(\tau_0 + 0)$ is a solution of problem (2.4) which, written out in detail, is

$$\begin{aligned} &-(h_0' + y'h)F(0)\dot{z}(\tau_0 + 0) \to \min_y \\ &-(h_0' + y'h)a(t) \geq 0,\ t \in T^*(\tau_0);\ -(h_0' + y'h)a(t) \leq 0,\ t \in T_*(\tau_0) \\ &y_i \geq 0,\ i \in M_a;\ y_i = 0,\quad i \in M \setminus M_a \end{aligned} \tag{4.1}$$

where

$$M_a = M_a(\tau_0),\quad h_i = \partial f_i(x_{\tau_0}(t^*))/\partial x,\quad i = 0,1,\ldots,m;\quad h = \left\| \begin{matrix} h_i' \\ i = 1,2,\ldots,m \end{matrix} \right\|$$

and $T^*(\tau)$, $T_*(\tau)$ are the closures of the sets $\{t \in T: u_\tau(t) = 1\}$, $\{t \in T: u_\tau(t) = -1\}$, respectively.

Put

$$\{l_j, j = 1,2,\ldots,r\} = \{t \in (0,t^*): u_{\tau_0}(t - 0) \neq u_{\tau_0}(t + 0)\}$$

$$m_* = \operatorname{rank} \left\| \begin{matrix} h_i'a(l_j),\ j = 1,2,\ldots,r \\ i \in M_a \end{matrix} \right\| \tag{4.2}$$

(We assume that $r = 0$ if $u_{\tau_0}(\cdot)$ has no points of discontinuity.)

We shall assume that the following assumption holds.

Assumption 1 (the analogue of Sleiter's condition for problem (4.1)). One of the following conditions holds: either (a) $m_* = |M_a|$, or (b) $m_* < |M_a|$ and no $\bar{t} \in T$ exists such that

$$(h_0' + y'h)a(\bar{t}) = (h_0' + y'h)\dot{a}(\bar{t}) = 0,\quad \forall y \in Y(\tau_0)$$

It follows from property 1 of the set $Y(\tau_0)$ that problem (4.1) has a solution. Let y^* be a solution of problem (4.1). Put

$$\begin{aligned} &\{t_j^*, j = 1,2,\ldots,p^*\} = \{t \in T: (h_0' + y^{*\prime}h)a(t) = 0\} \\ &t_j^* < t_{j+1}^*,\ j = 1,2,\ldots,p^* - 1 \\ &J^* = \{1,2,\ldots,p^*\},\ J_R^* = \{j \in J^*: t_j^* \in \{l_i, i = 1,2,\ldots,r\}\} \\ &M_a^* = \{i \in M_a: y_i^* > 0\},\ M_a^0 = M_a \setminus M_a^* \\ &\alpha_j = -u_{\tau_0}(t_j^* + 0)\ \text{ if }\ t_j^* \neq t^*;\ \alpha_j = u_{\tau_0}(t_j^* - 0)\ \text{ if }\ t_j^* = t^*;\ j \in J^* \end{aligned} \tag{4.3}$$

It follows from Assumption 1 and the optimality criterion of the plan y^* in problem (4.1) that numbers $\rho_j, j \in J^*$ exist for which

$$\rho_j \geq 0,\ j \in J^* \setminus J_R^*;\ \gamma_i = 0,\ i \in M_a^*;\ \gamma_i \leq 0,\ i \in M_a^0 \tag{4.4}$$

where

$$\gamma_i = h_i'(\sum_{j\in J^*} \rho_j \alpha_j a(t_j^*) + F(0)\dot{z}(\tau_0+0)),\ i \in M_a$$

We shall assume that the solution y^* of problem (4.1) is such that the following assumption holds.

Assumption 2

$$(h_0' + y^{*\prime} h)\ddot{a}(t_j^*) \neq 0 \quad \text{if} \quad t_j^* \in (0, t^*),\ j \in J^* \setminus J_R^*$$

$$(h_0' + y^{*\prime} h)\dot{a}(t_j^*) \neq 0 \quad \text{if} \quad t_j^* = 0 \vee t^* \ \text{or} \ j \in J_R^*$$

$$k_* = \operatorname{rank} \left\| \begin{matrix} h_i'a(t_j^*),\ j \in J^* \\ i \in M_a^* \end{matrix} \right\| = \min\{| M_a^* |, | J^* |\}$$

Construction of the structure $S(\tau + 0)$ *and the defining elements* $Q(\tau_0 + 0)$. We have the following possibilities: $k_* = |M_a^*| < |J^*|$ (case A) and $k_* = |J^*| \leqslant |M_a^*|$ (case B).

Consider case A. We introduce the notation

$$\tilde{D} = \operatorname{diag}(d(t_j^*),\ j \in J^*),\ d(t) = \left| \frac{\partial \psi'(t, y^*, x_{\tau_0}(t^*))b}{\partial t} \right|$$

$$D = D(x_{\tau_0}(t^*), y^*),\ D(x,y) = \frac{\partial^2 f_0(x)}{\partial x^2} + \sum_{s=1}^m y_s \frac{\partial^2 f_s(x)}{\partial x^2}$$

$$B = (B_{(j)} = 2\alpha_j a(t_j^*),\ j \in J^*),\ s = (s_j, j \in J^*)$$

and consider the quadratic programming problem

$$\begin{gathered} I(s) = [(F(0)\dot{z}(\tau_0+0) + Bs)' D (F(0)\dot{z}(\tau_0+0) + Bs) + s'\tilde{D}s]/2 \to \min_s \\ h_i'(Bs + F(0)\dot{z}(\tau_0+0)) = 0,\ i \in M_a^* \\ h_i'(Bs + F(0)\dot{z}(\tau_0+0)) \leqslant 0,\ i \in M_a^0;\ s_j \geqslant 0,\ j \in J^* \setminus J_R^* \end{gathered} \tag{4.5}$$

It is obvious from (4.4) that an admissible plan in problem (4.5) exists. The function $I(s)$ obviously has a lower limit. Consequently, problem (4.5) has a solution.

Let $s^* = (s_j^*, j \in J^*)$ be a solution of problem (4.5). Put

$$J_{(*)} = J_R^* \cup \{j \in J^* \setminus J_R^* : s_j^* \neq 0\},\ J_{(0)} = \{j \in J_{(*)} : d(t_j^*) = 0\} \tag{4.6}$$

$$M_{a*}^0 = \{i \in M_a^0 : h_i'(Bs^* + F(0)\dot{z}(\tau_0+0)) = 0\},\ \overline{M} = M_a^* \cup M_{a*}^0$$

By the optimality criterion, numbers ξ_i, $i \in \overline{M}$, $\xi_i \geqslant 0$, $i \in M_{a*}^0$ exist, such that the components of the vector

$$\Delta = (\Delta_j, j \in J^*) = B'(D(F(0)\dot{z}(\tau_0+0) + Bs^*) + \sum_{i\in\overline{M}} \xi_i h_i) + \tilde{D}s^*$$

satisfy the relations

$$\Delta_j = 0,\ j \in J_{(*)};\ \Delta_j \geqslant 0,\ j \in J^* \setminus J_{(*)}$$

We shall assume that the following conditions are satisfied; together with Assumption 2, they guarantee that the solutions y^*, s^* of problems (4.1) and (4.5), respectively, are unique.

Assumption 3. In case A

$$\Delta_j > 0,\ j \in J^* \setminus J_{(*)};\ \xi_i > 0,\ i \in M_{a*}^0;\ \det P_{(*)} \neq 0$$

where

$$P_{(*)} = \left\| \begin{matrix} \bar{H}B_{(*)} & 0 \\ R_{(*)} & B'_{(*)}\bar{H}' \end{matrix} \right\|, \quad \bar{H} = \left\| \begin{matrix} h'_i \\ i \in \bar{M} \end{matrix} \right\|$$

$$B_{(*)} = (B_{(j)}, j \in J_{(*)}), \quad R_{(*)} = B'_{(*)}DB_{(*)} + \mathrm{diag}(d(t^*_j), j \in J_{(*)})$$

Put

$$\bar{p} = |J_{(*)}| + |J_{(0)}|, \quad \bar{k} = -u_{\tau_0}(+0) \text{ if } \bar{t}_1 = 0, \quad \bar{k} = u_{\tau_0}(+0) \text{ if } \bar{t}_1 \neq 0$$
$$\{\bar{t}_j, j = 1, 2, \ldots, \bar{p}\} = \{t^*_j, j \in J_{(*)};\ t^*_j, j \in J_{(0)}\} \tag{4.7}$$
$$\bar{t}_j \leq \bar{t}_{j+1}, \quad j = 1, 2, \ldots, \bar{p} - 1$$

Theorem 1. Let τ_0 be a non-regular point and suppose Assumptions 1–3 hold for problems (4.1) and (4.5). Then

1) for $\tau = \tau_0 + 0$ the structure and defining elements of the solution have the form

$$S(\tau_0 + 0) = \{p(\tau_0 + 0) = \bar{p},\ k(\tau_0 + 0) = \bar{k},\ M_a(\tau_0 + 0) = \bar{M}$$
$$l_*(\tau_0 + 0) = l^*(\tau_0 + 0) = 0,\ M_0(\tau_0 + 0) = L(\tau_0 + 0) = \emptyset\} \tag{4.8}$$
$$Q(\tau_0 + 0) = \{t_j(\tau_0 + 0) = \bar{t}_j,\ j = 1, 2, \ldots, \bar{p};\ y(\tau_0 + 0) = y^*\}$$

2) for $\tau \in T^+(\tau_0)\backslash\tau_0$, problems OC($\tau$) have non-degenerate solutions whose structure and defining elements are uniquely defined by relations (3.1), where $p = \bar{p}$, $k = \bar{k}$, $M_* = \bar{M}$, $S = S(\tau_0 + 0)$, and the initial conditions $Q(\tau_0 + 0)$. An optimal control $u_\tau(\cdot)$ for $\tau \in T^+(\tau_0)\backslash\tau_0$ is constructed by the rules (3.3).

The scheme of the proof of Theorem 1 is similar to that of the theorem in [11].

Consider case B. Note that in this case problem (4.5) has a unique admissible plan $s^* = (s^*_j = \rho_j/2, j \in J^*)$. Consequently, it is the unique optimal plan of problem (4.5). Put

$$J_{(*)} = J^*, \quad J_{(0)} = \{j \in J_{(*)} : d(t^*_j) = 0\}, \quad \bar{M} = M^*_a \tag{4.9}$$

Then the conditions that guarantee the uniqueness of the solution y^* of problem (4.1) take the following form.

Assumption 3′. In case B

$$\gamma_i < 0,\ i \in M^0_a;\ \rho_j > 0,\ j \in J_{(*)} \setminus J^*_R$$

$$\mathrm{rank} \left\| \begin{matrix} h'_i a(t^*_j),\ j \in J_{(*)};\ h'_i \dot{a}(t^*_j),\ j \in J_{(0)} \\ i \in \bar{M} \end{matrix} \right\| = |\bar{M}|$$

We define the parameters $\bar{p}, \bar{k}, \bar{t}_j (j = 1, 2, \ldots, \bar{p})$ according to the rules (4.7). Theorem 1 holds for case B provided Assumption 3 is replaced by Assumption 3′.

Thus, if Assumptions 1–3(3′) are satisfied, Theorem 1 enables one to define solutions of OC(τ) uniquely in a right neighbourhood $T^+(\tau_0)$ of a non-regular point τ_0, given a solution of problem OC(τ_0). Similar results hold for the properties of the solutions of problems OC(τ) in a left neighbourhood $T^-(\tau_0)$ of the point τ_0.

5. THE DEPENDENCE OF THE PERFORMANCE CRITERION ON THE PARAMETER. THE EXTREMUM PROPERTY OF THE STRUCTURE $S(\tau_0 + 0)$

We have presented rules for constructing the vector $y(\tau_0 + 0)$ and structure $S(\tau_0 + 0)$: the vector $y(\tau_0 + 0)$ was determined by solving problem (4.1), and the structure $S(\tau_0 + 0)$ by solving problem (4.5). These rules may be given a different interpretation, related to the properties of the function

$$f_*(\tau)=f_0(x_\tau(t^*)),\ \tau\in[\tau_*,\tau^*] \tag{5.1}$$

We will now consider that interpretation. It is well known that under the assumptions made above concerning the functions $f_i(x), x\in R^n$ $(i=0,1,\dots,m)$; $z(\tau), \tau\in[\tau_*,\tau^*]$, function (5.1) will be continuous. Let us investigate its derivatives.

Let $\tau_0\in[\tau_*,\tau^*]$. It can be shown that

$$df_*(\tau_0+0)/d\tau=(h_0'+y'(\tau_0+0)h)F(0)\dot z(\tau_0+0) \tag{5.2}$$

Obviously, given a known solution $u_{\tau_0}(\cdot)$ of problem OC(τ_0) and a known vector $\dot z(\tau_0+0)$, the first derivative of function (5.1) at $\tau=\tau_0+0$ depends only on the choice of the vector $y(\tau_0+0)$ but not on the choice of the structure $S(\tau_0+0)$. Comparing relations (4.1) and (5.2), we conclude that the condition governing the choice of the vector $y(\tau_0+0)\in Y(\tau_0)$ is maximization of the first derivative of function (5.1) at the point $\tau=\tau_0+0$.

Suppose the vector $y(\tau_0+0)=y^*$ has been determined. We introduce the notation (4.2) and (4.3). Let $J_{(i)}, M_{(i)}$ $(i=1,2,\dots,\varepsilon)$ denote all possible pairs of subsets in $J^*\backslash J_R^*$ and M_a^0. We put

$$J_{(*)}^{(i)}=J_R^*\cup J_{(i)},\ J_{(0)}^{(i)}=\{j\in J_{(*)}^{(i)}:d(t_j^*)=0\},\ \overline{M}^{(i)}=M_a^*\cup M_{(i)}$$

and construct parameters $p^{(i)}, k^{(i)}, Q^{(i)}=\{t_j^{(i)}, j=1,2,\dots,p^{(i)}; y^*\}$ by rules (4.7), replacing $J_{(*)}, J_{(0)}, \overline{M}$ by $J_{(*)}^{(i)}, J_{(0)}^{(i)}, \overline{M}^{(i)}$. Now let E denote the set of indices $i\in\{1,\dots,\varepsilon\}$ for which continuous functions $t_j^{(i)}(\tau)$ $(j=1,2,\dots,p^{(i)})$, $y^{(i)}(\tau)$, $\tau\in T^+(\tau_0)$ exist satisfying conditions (3.1), where $p=p^{(i)}$, $k=k^{(i)}$, $M_*=\overline{M}^{(i)}$, initial conditions $\{t_j^{(i)}(\tau+0), j=1,2,\dots,p^{(i)}; y^{(i)}(\tau+0)\}=Q^{(i)}$, and the inequalities

$$\begin{gathered} t_j^{(i)}(\tau)\le t_{j+1}^{(i)}(\tau),\ j=1,2,\dots,p^{(i)}-1;\ t_1^{(i)}(\tau)\ge 0,\ t_{p^{(i)}}^{(i)}\le t^* \\ f_l(\mu(\tau,k^{(i)},\ t_j^{(i)}(\tau),\ j=1,2,\dots,p^{(i)}))\le 0,\ l\in M\setminus\overline{M}^{(i)},\ \tau\in T^+(\tau_0) \end{gathered} \tag{5.3}$$

It is obvious that the first three elements

$$S^*(\tau_0+0)=\{p(\tau_0+0),k(\tau_0+0),M_a(\tau_0+0)\}$$

of the desired structure $S(\tau+0)$ belong to the set of parameters

$$S^{*(i)}=\{p^{(i)},k^{(i)},\overline{M}^{(i)}\},\ i\in E$$

Note that, essentially, it is the parameters

$$S^*(\tau)=\{p(\tau),k(\tau),M_a(\tau)\}$$

that determine the structure $S(\tau)$ if no interval $[\bar\tau,\bar{\bar\tau}]\subset[\tau_*,\tau^*]$, $\bar\tau<\bar{\bar\tau}$ exists, where $\beta(\tau)>0$, $\tau\in[\bar\tau,\bar{\bar\tau}]$.

For $i\in E$, $\tau\in T^+(\tau_0)$, we construct controls $u_\tau^{(i)}(\cdot)$ as follows:

$$\begin{gathered} u_\tau^{(i)}(t)=(-1)^j k^{(i)},\ t\in[t_j^{(i)}(\tau),t_{j+1}^{(i)}(\tau)[,\ j=0,1,\dots,p^{(i)} \\ t_0^{(i)}(\tau)=0,\ t_{p^{(i)}+1}^{(i)}(\tau)=t^* \end{gathered} \tag{5.4}$$

It follows from relations (3.1) and (5.3) that the controls $u_\tau^{(i)}(\cdot), i\in E$ are admissible in problem OC(τ) for $\tau\in T^+(\tau_0)$.

Let $x_\tau^{(i)}(\cdot)$ be a trajectory corresponding to a control $u_\tau^{(i)}(\cdot)$ and initial state $z(\tau)$. Consider the functions

$$f_*^{(i)}(\tau)=f_0(x_\tau^{(i)}(t^*)),\ \tau\in T^*(\tau_0),\ i\in E$$

It is obvious that

$$f_*^{(i)}(\tau_0+0) = f_*(\tau_0+0) = f_0(x_{\tau_0}(t^*)) = \text{const}$$

$$df_*^{(i)}(\tau_0+0)/d\tau = df_*(\tau_0+0)/d\tau = (h_0' + y^{*'}h)F(0)\dot{z}(\tau_0+0) = \text{const}, \quad i \in E$$

Let us evaluate $d^2f_*^{(i)}(\tau)/d\tau^2$. To that end, we introduce vectors $s^{(i)} = (s_j^{(i)}, j \in J^*)$

$$\begin{aligned} s_j^{(i)} &= \dot{t}_{q(j)+1}^{(i)}(\tau_0) - \dot{t}_{q(j)}^{(i)}(\tau_0), \quad j \in J_{(0)}^{(i)} \\ s_j^{(i)} &= \dot{t}_{q(j)}^{(i)}(\tau_0), \quad j \in J_{(*)}^{(i)} \setminus (J_{(0)}^{(i)} \cup J_{(k)}^{(i)}) \\ s_j^{(i)} &= -\dot{t}_{q(j)}^{(i)}(\tau_0), \quad j \in J_{(k)}^{(i)} = \{j \in J_{(*)}^{(i)} : t_j^* = t^*\}; \quad s_j^{(i)} = 0, \quad j \in J^* \setminus J_{(*)}^{(i)} \end{aligned} \tag{5.5}$$

where $q(j)$ is an index from $\{1, \dots, p^{(i)}\}$ such that

$$t_{q(j)-1}^{(i)} \neq t_{q(j)}^{(i)} = t_j^*, \quad j \in J_{(*)}^{(i)}$$

It can be shown that

$$d^2 f_*^{(i)}(\tau_0+0)/d\tau^2 = 2I(s^{(i)}) \tag{5.6}$$

It follows from relations (3.1) and (3.5) that the components of the vector $s^{(i)}$ (5.5) satisfy the restrictions of problem (4.5).

Thus, if $i \in E$, there is for every set of parameters $S^{*(i)}$ a plan $s^{(i)}$ of problem (4.5), and twice the value of the objective function $I(s)$ for this plan equals the value at $\tau = \tau_0 = 0$ of the second derivative of the objective function of problems $OC(\tau)$, $\tau \in T^+(\tau_0)$, evaluated along the controls $u_\tau^{(i)}(\cdot)$ (5.4) generated by the set of parameters $S^{*(i)}$.

By construction, corresponding to the parameters $S^*(\tau_0 + 0) = S^{*(i_0)}$ we have a vector $s^* = s^{(i_0)}$ which is a solution of problem (4.5). Consequently

$$d^2 f_*^{(i)}(\tau_0+0)/d\tau^2 \,|_{\tau=\tau_0+0} \geq d^2 f_*^{(i_0)}(\tau_0+0)/d\tau^2 \,|_{\tau=\tau_0+0}, \quad i \in E$$

Hence it is obvious that, using the rules described above for constructing the elements $S^*(\tau_0 + 0)$ of the structure $S(\tau_0 + 0)$, one can give the following extremum interpretation: the elements $S^*(\tau_0 + 0)$ of the structure $S(\tau_0 + 0)$ belong to the set of parameters $S^{*(i)}$, $i \in E$, at which the following minimum is reached

$$\min_{i \in E} d^2 f_*^{(i)}(\tau_0+0)/d\tau^2 \tag{5.7}$$

It is obvious that if the minimum in this expression is reached at a unique $i_0 \in E$, then the elements $S^*(\tau_0 + 0)$ of the structure $S(\tau_0 + 0)$ are uniquely defined.

We note that the conditions formulated in Assumption 1–3 (3′) guarantee that

1) y^* is the unique vector at which the first derivative of function (5.1) attains its maximum value for $y \in Y(\tau_0)$;

2) the maximum in (5.7) is reached at a unique index $i_0 \in E$;

3) we have

$$l_*(\tau_0+0) = l^*(\tau_0+0) = 0; \quad L(\tau_0+0) = M_0(\tau_0+0) = \emptyset$$

Thus, if Assumptions 1–3 (3′) are satisfied, the vector $y(\tau_0 + 0)$ and the structure $S(\tau_0 + 0)$ are uniquely constructed by the rules described above.

If the conditions of Assumptions 1–3 (3′) are violated, the rules as described do not guarantee the unique construction of $y(\tau_0 + 0)$ and $S(\tau_0 + 0)$. It should be noted, however, that the probability of such situations actually occurring is very small [13].

6. THE METHOD OF CONTINUATION OF SOLUTIONS WITH RESPECT TO A PARAMETER

Using the scheme of the standard method of continuation with respect to a parameter [14, 15], the special method for solving systems of non-linear problems in the neighbourhood of a non-regular point [14, 15], and the rules derived above for constructing a new structure and new defining elements at a non-regular point, one can easily devise a numerical method for constructing solutions $u_\tau(\cdot)$ of problem OC(τ) for all parameter values $\tau \in [\tau_*, \tau^*]$. We will outline the general scheme of the method.

Let us assume that at a time $\tau = \tau_*$ there is a known optimal control (OC) $u_{\tau_*}(\cdot)$ of problem OC(τ_*), its structure $S(\tau_*)$ and defining elements $Q(\tau_*)$.

If τ_* is a regular point, we put

$$\tau_0 := \tau_*, \quad p := p(\tau_*), \quad k := k(\tau_*), \quad M_* := M_a(\tau_*)$$

$$\bar{t}_j := t_j(\tau_*)(j = 1, 2, \ldots, p), \quad \bar{y} := y(\tau_*)$$

and proceed to operations A^*. If τ_* is a non-regular point, then, putting $\tau_0 := \tau_*$, we proceed to operations B^*.

Operations A^*. Let the known parameters be $\tau_0, p, k, M_*; \bar{t}_j (j = 1, 2, \ldots, p), \bar{y}$.

For $\tau \in [\tau_0, \tau_1]$, OCs $u_\tau(\cdot)$ of problem OC(τ) are constructed according to rules (3.3), with the switching times $t_j(\tau)$ ($j = 1, 2, \ldots, p$) and the vector $y(\tau)$ uniquely determined from relations (3.1) and the initial conditions

$$t_j(\tau_0 + 0) = \bar{t}_j (j = 1, 2, \ldots, p); \quad y(\tau_0 + 0) = \bar{y}$$

where $\tau_1 = \min\{\tau^*, \bar{\tau}_1\}$, $\bar{\tau}_1$ being the non-regular parameter value closest to τ_0 on the right. If $\tau_1 = \tau^*$, the algorithm ends its operation. If $\tau_1 < \tau^*$, we put $\tau_0 := \tau_1$ and proceed to operations B^*.

Operations B^*. Suppose for the non-regular point $\tau = \tau_0$ we known an OC $u_{\tau_0}(\cdot)$ of problem OC(τ_0), its structure $S(\tau_0)$ and defining elements $Q(\tau_0) = \{t_j(\tau_0), j = 1, 2, \ldots, p(\tau_0); y(\tau_0)\}$. Using this information, we formulate and solve problem (4.1). The vector $y(\tau_0)$ may be used as the initial plan.

Suppose Assumption 1 is satisfied. The solution of problem (4.1) yields the construction of an optimal plan y^* and of numbers $\rho_j, j \in J^*$, satisfying (4.4).

Note that if $m_* = |M_a|$, the vector $y(\tau_0)$ is unique and is therefore an optimal plan of problem (4.1). Thus, in that case there is no need to solve problem (4.1).

Suppose the vector y^* satisfies Assumption 2. Formulate and solve problem (4.5). The numbers $\rho_j/2$, $j \in J^*$, may be taken as the initial plan. Note that in case B (see Section 5) there is no need to solve problem (4.5).

Let s^* denote an optimal plan of problem (4.5). Suppose Assumption 3 (3′) holds. Putting $\bar{y} = y^*$, define the set $\bar{M}$ by rule (4.6) (4.9) and parameters $\bar{p}, \bar{k}, \bar{t}_j (j = 1, 2, \ldots, \bar{p})$ by rules (4.7). Using these parameters, proceed to operations A^*, putting $p = \bar{p}, k = \bar{k}, M_* = \bar{M}$.

Remarks. 1. The dimensionalities of problems (4.1) and (4.5) depend on the degree of non-regularity of the point τ_0: the larger $\beta(\tau_0)$, the more dimensions problems (4.1) and (4.5) will have. For example, if $\beta(\tau_0) = 1$, one of problems (4.1) and (4.5) has a unique plan (dimension 0), while the other has a plan and only one admissible direction in which one can move from the plan (dimension 1).

2. If τ is a regular point, the Jacobian of Eqs (3.1) is non-singular. In that case, in order to find the functions $t_j(\tau)$ ($j = 1, 2, \ldots, p$), $y(\tau)$ from (3.1), one can use any standard method for continuing the solutions of systems of non-linear equations [15]. If τ is a non-regular point, the Jacobian of Eqs (3.1) may turn out to be singular (when $L(\tau) \neq \emptyset$). In that situation the standard methods do not work well, and special methods, such as those of [16], are needed to solve Eqs (3.1) in the neighbourhood of a non-regular point.

7. EXAMPLE

We will illustrate the application of the scheme proposed above for constructing solutions of problems OC(τ), $\tau \in [\tau_*, \tau^*]$, by a simple example. Consider the following family of OC problems

$$\mathrm{OC}_*(\tau): \begin{cases} x'(3)x(3)/2 \to \min \\ \dot{x} = Ax + bu, \quad x(0) = z(\tau), \quad |u(t)| \leq 1, \quad t \in [0, 3] \\ f_1(x(3)) := x_4(3) - 0.5 \leq 0 \\ f_2(x(3)) := -x_4(3) - 1.5 \leq 0 \end{cases}$$

where

$$x' = (x_1,\ x_2,\ x_3,\ x_4),\quad z'(\tau) = (-19.125 + 27\tau,\quad 2.5 - 45.5\tau,\quad 3.5 + 36\tau,\quad -3 - 5\tau)$$

$$b = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Vmatrix},\quad A = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{Vmatrix},\quad \tau \in [\tau_* = -0.1,\ \tau^* = 0.1]$$

Suppose that when $\tau = \tau_*$ there is a known solution of problem $OC_*(\tau_*)$: the OC has the form

$$u_\tau(t) = 1,\quad t \in [0,\ t_1(\tau)[;\quad u_\tau(t) = -1,\quad t \in [t_1(\tau),\ 3] \tag{7.1}$$

where $\tau = \tau_*$, $t_1(\tau_*) = 2.902$. Corresponding to this control are the following structures and defining elements

$$S(\tau_*) = \{p(\tau_*) = 1,\quad k(\tau_*) = 1,\quad M_a(\tau_*) = \emptyset,\quad l_*(\tau_*) = l^*(\tau_*) = 0,\quad M_0(\tau_*) = L(\tau_*) = \emptyset\}$$

$$Q(\tau_*) = \{t_1(\tau_*) = 2.902;\quad y(\tau_*) = (y_1(\tau_*) = 0,\quad y_2(\tau_*) = 0)\}$$

The solution of the problem $OC_*(\tau_*)$ is non-degenerate, since $\beta(\tau_*) = 0$. According to what was stated in Section 3, if $\tau \in T^+(\tau_*)$, problems $OC_*(\tau)$ have a non-degenerate structure $S(\tau) = S(\tau_*)$ and an OC of the form (7.1), whee the time $t_1(\tau)$ is uniquely defined by relations (3.1), which in this example have the form

$$\left(F(0)z(\tau) + \int_0^{t_1(\tau)} a(t)dt - \int_{t_1(\tau)}^{3} a(t)dt\right)' a(t_1(\tau)) = 0 \tag{7.2}$$

where

$$a(t) = \begin{Vmatrix} (3-t)^3/6 \\ (3-t)^2/2 \\ (3-t) \\ 1 \end{Vmatrix},\quad F(0) = \begin{Vmatrix} 1 & 3 & 9/2 & 9/2 \\ 0 & 1 & 3 & 9/2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

By construction, $u_\tau(t) = \operatorname{sign} \Delta_\tau(t)$, $t \in [0, 3]$, where

$$\Delta_\tau(t) = \psi'(t,\ y(\tau) = 0,\ x_\tau(3))b,\quad t \in [0,\ 3] \tag{7.3}$$

Figure 1 shows a graph of the function $\Delta_\tau(t)$, $t \in [0, 3]$, for $\tau = -0.05$.

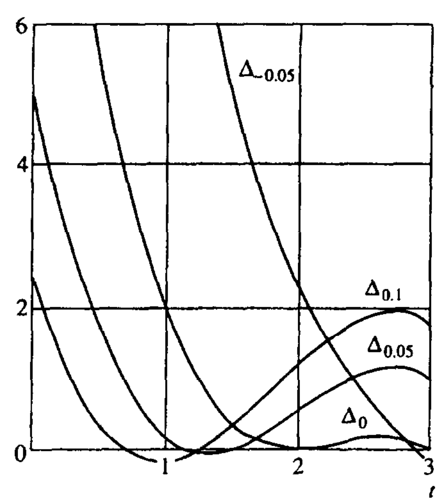

Fig. 1

According to Section 6, OCs of problems $OC_*(\tau)$ are constructed by rules (7.1), (7.2) for $\tau \in [\tau_*, \tau_1]$, where $\tau_1 > \tau_*$ is the parameter value at which the solution becomes degenerate. In the example considered here, degeneracy occurs at $\tau = \tau_1 = 0$:

$$S(-0) = S(\tau_*), \quad Q(-0) = \{t_1(-0) = 3, \quad y(-0) = 0\}$$

$$S(0) = \{p(0) = 2, \quad k(0) = 1, \quad M_a(0) = \emptyset, \quad l_*(0) = 0, \quad l^*(0) = 1, \quad M_0(0) = \emptyset, \quad L(0) = \{1\}\}$$

$$Q(0) = \{t_1(0) = 2, \quad t_2(0) = 3; \quad y(0) = 0\}$$

The degree of degeneracy is $\beta(0) = 2$. A graph of the function $\Delta_\tau(t)$ for $\tau = 0$ is shown in the figure: $u_\tau(t) = 1$.

To construct a new structure $S(+0)$ and new defining elements $Q(+0)$, we use the results of Section 4.

Since there are no active terminal constraints at $\tau = 0$ (that is, $M_a(0) = \varnothing$), it clearly follows that $y(\tau) \equiv 0$ for $\tau \in T^+(0)$ and there is no need to solve problem (4.1) in order to find $y(+0) = y^* = 0$.

To construct a new structure $S(+0)$, we formulate the quadratic programming problem (4.5), which is in this case

$$(-82, -10)s + 0.5s' \begin{Vmatrix} 82/9, & -4 \\ -4 & 5 \end{Vmatrix} s \to \min_{s=(s_1, s_2)}, \quad s_1 \geqslant 0, \quad s_2 \leqslant 0$$

and the solution is

$$s^0 = (s_1^0 = 9, \quad s_2^0 = 0)$$

Assumptions 1–3 are satisfied, and therefore, by Theorem 1, new structures $S(+0)$ and defining elements $Q(+0)$ are constructed by rules (4.6)–(4.8); they take the form

$$S(+0) = \{p(+0) = 2, \quad k(+0) = 1, \quad M_a(+0) = \emptyset, \quad l_*(+0) = l^*(+0) = 0, \quad M_0(+0) = L(+0) = \emptyset\}$$

$$Q(+0) = \{t_1(+0) = 2, \quad t_2(+0) = 2; \quad y(+0) = 0\}$$

For $\tau \in T^+(0)$, and OC is constructed by the rule

$$u_\tau(t) = 1, \quad t \in [0, \ t_1(\tau)[\cup[t_2(\tau), \ 3]; \quad u_\tau(t) = -1, \quad t \in [t_1(\tau), \ t_2(\tau)] \tag{7.4}$$

The times $t_1(\tau)$ and $t_2(\tau)$ are uniquely found from relations (3.1), where

$$p = p(+0) = 2, \quad k = k(+0) = 1, \quad M_* = M_a(+0) = \emptyset$$

and by the initial conditions $Q(+0)$. In the example in question, these relations have the form

$$\left(F(0)z(\tau) + \int_0^{t_1(\tau)} a(t)dt - \int_{t_1(\tau)}^{t_2(\tau)} a(t)dt + \int_{t_2(\tau)}^{3} a(t)dt \right)' a(t_i(\tau)) = 0, \quad i = 1, 2 \tag{7.5}$$

As τ varies from $+0$ to $\tau^* = 0.1$ the solution does not degenerate. Consequently, for all $\tau \in [0, \tau^*]$, solutions of problems $OC(\tau_*)$ are constructed by rule (7.4), where the times $t_1(\tau)$ and $t_2(\tau)$ are uniquely determined from (7.5).

By construction, $u_\tau(t) = \operatorname{sign} \Delta_\tau(t)$, $t \in [0, 3]$. Graphs of the functions $\Delta_\tau(t)$, $t \in T$, (7.3) for $\tau = 0.05$ and $\tau = 0.1$, are shown in the figure.

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